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Optimal replacement policy for obsolete components with general failure rates

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SUMMARY

Identical components are considered, which become obsolete once new-type ones are available, more reliable and less energy consuming. We envision different possible replacement strategies for the old-type components by the new-type ones: either purely preventive, where all old-type components are replaced as soon as the new-type ones are available; either purely corrective, where the old-type ones are replaced by new-type ones only at failure; or a mixture of both strategies, where the old-type ones are first replaced at failure by new-type ones and next simultaneously preventively replaced after a fixed number of failed old-type components.

To evaluate the respective value of each possible strategy, a cost function is considered, which represents the mean total cost on some finite time interval $[0, t]$. This function takes into account replacement costs, with economical dependence between simultaneous replacements, and also some energy consumption (and/or production) cost, with a constant rate per unit time.

A full analytical expression is provided for the cost function induced by each possible replacement strategy. The optimal strategy is derived in long-time run. Numerical experiments conclude the paper.

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1. INTRODUCTION

We here consider identical and independent components, which may be part of a single industrial equipment or dispatched in different locations, indifferently. Those components are degrading with time, and their random lifetimes follow some common general distribution. We assume that at some

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fixed time, say time 0, some new components appear in the market, issued from a new technology, which makes them more reliable, less energy consuming and more performing. Such new-type of components may be substituted to the older ones with no problem of compatibility. There is no stocking of old-type components and after time 0, no old-type component is available anymore (or the industrialist is not allowed to use old-type components anymore, e.g. for safety reasons). After time 0, any failed component, either old-type or new-type, is then instantaneously replaced by a new-type one. At time 0, each old-type component is in use since some random time, with some random remaining lifetime. If the new-type components are much less energy consuming than the older ones and if the period of interest is very long, it may then be expedient to remove all old-type components immediately at time 0 and replace them by new-type ones, leading to some so-called purely preventive replacement strategies. On the contrary, in case there is no much improvement between both technologies and if the period of interest is short, it may be better to wait until the successive failures of the old-type components and replace them by new-type ones only at failure, leading to some purely corrective replacement strategy. More generally, some mixture of both strategies, preventive and corrective, may also be envisioned and may lead to lower costs, as can be seen later. The point of the present paper is to look for the optimal replacement strategy among the purely preventive one, the purely corrective one and the mixtures of both strategies envisioned here. To evaluate the respective value of each possible strategy, a cost function is considered, which represents the mean total cost on some finite time interval $[0, t]$. This function takes into account replacement costs, with economical dependence between simultaneous replacements (see [1]), and also some energy consumption (and/or production) cost, with a constant rate per unit time.

Looking at the published literature, it seems that a similar model as ours has not been considered very often yet. Indeed, lots of papers dealing with obsolescence are mainly concerned with problems of stock sizing and/or inventory policy (see [2–7], and references therein). In such papers, different changes in technology are envisioned, which may arrive randomly, with a degree of innovation, which may be random too. The demand for removing items from the stock may arrive at a constant rate or randomly, with eventual random size too. The problem is then to optimize the stock, which should not be too high because of eventual depreciation of stocked components due to obsolescence, but which should not be too low either, because of eventual shortage inducing additional costs. Such a problem is clearly very different from ours.

Other papers deal with some problem perhaps a little closer to ours ([8–15], and references therein): technological breakthroughs are considered in such papers, which may be random, both for their arrivals and/or for their degrees. The items usually degrade deterministically or even do not degrade at all. The point in those papers is firstly to model the technological breakthroughs, which is made diversely, and secondly to find an optimal replacement strategy in such an evolutionary context, which evolves independently of the studied items. In the present paper, a single technological breakthrough is considered, which is known to occur at time 0. The improvement due to the technological change is also known with certainty. However, the components here degrade stochastically with some general failure distribution. Also, an economic dependence is introduced in the case of simultaneous replacements (details further). Our model hence appears as simpler for the model of technological breakthrough but more sophisticated for the degradation of the items and for the cost function.

A model taking into account some stochastic deterioration as well as technological change may be found in [16]: an item is considered that degrades according to some discrete time Markov chain. A single technological change is considered, which occurs randomly. Using minimizing

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technics from [17], an algorithm is developed to choose between two possible decisions: keep or replace the item at initial time, according to some planned horizon. Here again, the model, as well as the method and results, is very different from ours.

A similar model as here may, however, be found in [18, 19] in the case of constant failure rates for both old-type and new-type components. Contrary to the present paper, all costs were beside discounted at time 0 in both papers. In such a context, it had been proved in [19] that in the case of constant failure rates, the only possible optimal strategies were either purely corrective or nearly purely preventive (details further), leading to some simple dichotomous decision rule.

A first attempt to note whether such a dichotomy is still valid in the case of general failure rates was done in [20] by Monte Carlo (MC) simulations. However, the length of the MC simulations did not allow to cover a sufficient range for the different parameters, making the answer difficult. Similarly, recent works [21, 22] proposed complex models including the present one, which are evaluated by MC simulations. Here again, the length of the MC simulations added to the complexity of the model does not allow to perform optimization on the replacement strategies.

The point of the present paper is hence to answer the following questions: Is the dichotomy proved in the case of constant failure rates still valid in the case of general failure rates? If not (and it will not), what are the possible optimal strategies? Finally, how can we find the optimal strategy?

This paper is organized as follows: the model is specified in Section 2. Section 3 is devoted to the analytical computation of the cost function entailed by each possible strategy on some fixed finite horizon. The optimal strategy is derived for long-time run in Section 4. In particular, it is analytically proved that the result from [19] is not valid anymore in the case of general failure distributions: any envisioned strategy may here be optimal with respect to the cost function. Numerical experiments are lead on in Section 5, validating the results of Section 4 and showing their interest even in the case of finite horizon. Concluding remarks end this paper in Section 6.

2. THE MODEL—NOTATIONS AND ASSUMPTIONS

We consider n identical and independent old-type components ($n \geqslant 2$). At time 0, such old-type components are up, in activity. For each $i = 1, \ldots, n$, the residual lifetime for the ith component is assumed to be some absolutely continuous random variable (r.v.) U_i, where all U_i's are not necessarily identically distributed. This means that the ith component is assumed to fail at time U_i. The successive times to failure of the n old-type components then appear as the order statistics of $(U_1, \ldots, U_n)$. They are denoted by $(U_{1:n}, \ldots, U_{n:n})$ in the following, where $U_{1:n} < \cdots < U_{n:n}$ almost everywhere (a.e.) because U_i's admit density with respect to Lebesgue measure.

After time 0, any failed component, either old-type or new-type, is always instantaneously replaced by a new-type one. All preventive replacements (by new-type components) are also instantaneous.

The following replacement strategies are envisioned:

- *Strategy 0*: The n old-type components are immediately replaced by n new-type ones at time 0. This is a purely preventive strategy. After time 0, there are exactly n new-type components.
- *Strategy 1*: No replacement is performed before the first failure, which occurs at time $U_{1:n}$. At time $U_{1:n}$, the failed component is correctively replaced and the $n-1$ non-failed old-type

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components are simultaneously preventively replaced. This is hence a nearly purely preventive strategy. Before time $U_{1:n}$, there are exactly n old-type components. After time $U_{1:n}$, there are exactly n new-type components.

- *Strategy K* ($1\leqslant K\leqslant n$): No preventive replacement is performed before the Kth failure, which occurs at time $U_{K:n}$. This means that only corrective replacements are performed up to time $U_{K:n}$ (at times $U_{1:n},\ldots,U_{K-1:n}$). At time $U_{K:n}$, the failed component is correctively replaced and the $n-K$ non-failed old-type components are simultaneously preventively replaced. Before time $U_{1:n}$, there are exactly n old-type components. After time $U_{K:n}$, there are exactly n new-type components. For $K\geqslant 2$, between times $U_{i:n}$ and $U^-_{i+1:n}$ ($1\leqslant i\leqslant K-1$), there are i new-type components and $n-i$ old-type ones.
- *Strategy n*: No preventive replacement is performed at all. Before time $U_{1:n}$, there are exactly n old-type components. Between times $U_{i:n}$ and $U^-_{i+1:n}$ ($1\leqslant i\leqslant n-1$), there are i new-type components and $n-i$ old-type ones. After time $U_{n:n}$, there are exactly n new-type components.

Once a new-type component is put into activity at time 0 or at time say $U_{i:n}$, it is next instantaneously replaced at failure by another new-type component. The successive lifetimes of such components are assumed to form a renewal process with eventual delay $U_{i:n}$; the i.i.d. inter-arrival times are distributed as some r.v. V with $\mathbb{P}(0\leqslant V<\infty)=1$ and $\mathbb{P}(V>0)>0$. The inter-arrival times are then non-negative and not identically 0. The renewal function associated with the non-delayed process is then finite on $\mathbb{R}_+$. It is denoted by ρ_V with

$$\rho_V(t)=\mathbb{E}(N_V([0,t]))=\mathbb{E}\left(\sum_{k\in\mathbb{N}^*}\mathbf{1}_{\{V^{(1)}+\cdots+V^{(k)}\leqslant t\}}\right)$$

where $V^{(1)},\ldots,V^{(k)},\ldots$ are the successive inter-arrival times and if I is an interval with $I\subset\mathbb{R}_+$, $N_V(I)$ stands for the number of renewals in I.

In the case of a delayed renewal process with $U_{i:n}$ for delay and inter-arrival times still distributed as V, the number of renewals on I is denoted by $N_{U_{i:n},V}(I)$.

To evaluate the respective value of each possible strategy, a cost function is considered, which represents the mean total cost on some time interval $[0,t]$. It is denoted by $C_K([0,t])$ when strategy K is used. Two types of costs are considered, with respective means $\mathrm{CR}_K([0,t])$ and $\mathrm{CE}_K([0,t])$. The first cost $\mathrm{CR}_K([0,t])$ corresponds to replacement costs: each solicitation of the repair team is assumed to entail a fixed cost r ($r\geqslant 0$). Each corrective and preventive replacement involves a supplementary cost, respectively, c_f and c_p, to be added to r ($0<c_p\leqslant c_f$). For instance, the cost for preventive replacement of i units ($0\leqslant i\leqslant n-1$), which comes along with the corrective replacement of one unit, is $r+c_f+ic_p$. As for the other cost (or benefit) $\mathrm{CE}_K([0,t])$, we assume a constant rate per unit time, with a higher rate for an old-type unit ($\eta+v$ with $v\geqslant 0$, $\eta\in\mathbb{R}$) than for a new-type (η). Such a rate may include energy consumption and/or production rate per unit time, e.g. The corresponding cost is called 'energy consumption cost' in the following. The 'energy consumption' cost for j new-type units and k old-type units on $[t_1,t_2]$ is $(j\eta+k(\eta+v))\ (t_2-t_1)$ with $0\leqslant t_1\leqslant t_2$ and $j+k=n$.

All components, both new-type and old-type, are assumed to be independent of each other.

In this paper, if X is a non-negative r.v., its cumulative density function (c.d.f.) is denoted by F_X, its survival function by $\bar{F}_X$ with $\bar{F}_X=1-F_X$ and its eventual probability density function (p.d.f.) by f_X. For $t\in\mathbb{R}_+$, we also set $X^t=\min(X,t)$.

Finally, we shall use the following notations:

$$a = \frac{r + c_f}{c_p} \geqslant 1$$

$$b = \frac{v}{c_p} \geqslant 0$$

and

$$x^+ = \max(x, 0)$$

for any real x.

3. COST FUNCTIONS

We first compute the mean cost on $[0, t]$ induced by strategy 0. Note contrary to strategy 1 or n, strategy 0 is not a special case of general strategy K.

Proposition 1
The mean cost on $[0, t]$ induced by strategy 0 is

$$C_0([0, t]) = n\eta t + r + nc_p(1 + a\rho_V(t))$$

for all $t \geqslant 0$.

Proof
The proof is very similar to that from [19], which we recall here for sake of completeness: when strategy 0 is used, the cost on $[0, t]$ is due to:

- the energy consumption of the n new-type units on $[0, t]$, which is equal to

$$\mathrm{CE}_0([0, t]) = n\eta t$$

- the preventive replacement of n components at time 0, which is equal to $nc_p + r$;
- the corrective replacements of the new-type units (among n), which fail on $[0, t]$: $n(r + c_f)\rho_V(t) = nc_p a\rho_V(t)$.

Hence the result. □

We now come to the general case with $1 \leqslant K \leqslant n$ and we first consider the mean *energy consumption* costs.

Lemma 2
For $1 \leqslant K \leqslant n - 1$,

$$\begin{aligned}\mathrm{CE}_{K+1}([0, t]) - \mathrm{CE}_K([0, t]) &= (n - K)v\mathbb{E}(U^t_{K+1:n} - U^t_{K:n}) \\ &= (n - K)c_p b\mathbb{E}(U^t_{K+1:n} - U^t_{K:n})\end{aligned}$$

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and for $1\leqslant K\leqslant n$,

$$\mathrm{CE}_K([0,t]) = v(n-K)\mathbb{E}(U_{K:n}^t) + v\sum_{i=1}^{K}\mathbb{E}(U_{i:n}^t) + n\eta t$$

Proof
We may first note that the mixture of both types of components is the same before time $U_{K:n}$ and after time $U_{K+1:n}$ for both strategies K and $K+1$. This implies that the difference of costs on $[0,t]$ between these two strategies is null when $t<U_{K:n}$. We then restrict the study to the case $t\geqslant U_{K:n}$ and calculate the difference of costs on $[U_{K:n}, U_{K+1:n}^t)$ where we recall that $U_{K+1:n}^t = \min(U_{K+1:n}, t)$.

When $t\geqslant U_{K:n}$, there are n new-type components when strategy K is applied, whereas there are K new-type and $n-K$ old-type ones for strategy $K+1$ on $[U_{K:n}, U_{K+1:n}^t)$. Consequently, the corresponding difference of energy consumption costs is due to the difference in energy consumption on $[U_{K:n}, U_{K+1:n}^t)$ of $n-K$ components, which are old-type in strategy $K+1$, and new-type in strategy K. We obtain

$$\begin{aligned}\mathrm{CE}_{K+1}([0,t]) - \mathrm{CE}_K([0,t]) &= (n-K)v\mathbb{E}((U_{K+1:n}^t - U_{K:n})\mathbf{1}_{\{U_{K:n}\leqslant t\}})\\ &= (n-K)v\mathbb{E}(U_{K+1:n}^t - U_{K:n}^t)\end{aligned}$$

and the first result.

Let us now compute $\mathrm{CE}_1([0,t])$: when strategy 1 is used, there are n old-type components up to $U_{1:n}^t$ and n new-type components on $(U_{1:n}^t, t]$ (eventually empty if $t\leqslant U_{1:n}$). We easily derive

$$\begin{aligned}\mathrm{CE}_1([0,t]) &= \mathbb{E}(n(\eta+v)U_{1:n}^t) + \mathbb{E}(n\eta(t-U_{1:n}^t))\\ &= nv\mathbb{E}(U_{1:n}^t) + n\eta t\end{aligned}$$

and the second result using

$$\mathrm{CE}_K([0,t]) = \sum_{i=1}^{K-1}(\mathrm{CE}_{i+1}([0,t]) - \mathrm{CE}_i([0,t])) + \mathrm{CE}_1([0,t])$$

and the reduction. □

We now look at the mean *replacement* costs.

Lemma 3
For $1\leqslant K\leqslant n$,

$$\begin{aligned}\mathrm{CR}_K([0,t]) = {}&(r+c_f)\sum_{i=1}^{K}[F_{U_{i:n}}(t) + \mathbb{E}(\rho_V((t-U_{i:n})^+))]\\ &+(n-K)[c_p F_{U_{K:n}}(t) + (r+c_f)\mathbb{E}(\rho_V((t-U_{K:n})^+))]\end{aligned}$$

and for $1\leqslant K\leqslant n-1$,

$$\begin{aligned}\frac{1}{c_p}(\mathrm{CR}_{K+1}([0,t]) - \mathrm{CR}_K([0,t])) = {}&(a-1)F_{U_{K+1:n}}(t) + a(n-K)[\mathbb{E}(\rho_V((t-U_{K+1:n})^+))\\ &-\mathbb{E}(\rho_V((t-U_{K:n})^+))] + (n-K)[F_{U_{K+1:n}}(t) - F_{U_{K:n}}(t)]\end{aligned}$$

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Proof
The replacement costs induced by strategy K are due to:

- eventual corrective replacement at $U_{1:n}$ (if $U_{1:n}\leqslant t$), ..., at $U_{K-1:n}$ (if $U_{K-1:n}\leqslant t$) of old-type components, and corrective replacements of the new-type components eventually put into activity at $U_{1:n},\dots,U_{K-1:n}$;
- if $t\geqslant U_{K:n}$: the $n-K$ preventive replacements at time $U_{K:n}$ and a corrective replacement;
- if $t\geqslant U_{K:n}$: the corrective replacements of the $n-K+1$ new-type components put into activity at $U_{K:n}$.

We obtain

$$\begin{aligned}\mathrm{CR}_K([0,t]) = (r+c_f)&\left(\sum_{i=1}^{K-1}\mathbb{P}(U_{i:n}\leqslant t)+\sum_{i=1}^{K-1}\mathbb{E}(N_{U_{i:n},V}([U_{i:n},t])\mathbf{1}_{\{U_{i:n}\leqslant t\}})\right)\\ &+(r+c_f+(n-K)c_p)\mathbb{P}(U_{K:n}\leqslant t)\\ &+(r+c_f)(n-K+1)\mathbb{E}(N_{U_{K:n},V}([U_{K:n},t])\mathbf{1}_{\{U_{K:n}\leqslant t\}})\end{aligned} \tag{1}$$

We may now express

$$\mathbb{E}(N_{U_{i:n},V}([U_{i:n},t])\mathbf{1}_{\{U_{i:n}\leqslant t\}})=\mathbb{E}(g(U_{i:n})) \tag{2}$$

where

$$g(U_{i:n}):=\mathbb{E}(N_{U_{i:n},V}([U_{i:n},t])\mathbf{1}_{\{U_{i:n}\leqslant t\}}|U_{i:n})$$

and $\mathbb{E}(\dots|U_{i:n})$ stands for the conditional expectation given $U_{i:n}$.

With abusive but, however, usual notation, we have for all $u\in\mathbb{R}_+$:

$$\begin{aligned}g(u) &= \mathbb{E}(N_{U_{i:n},V}([U_{i:n},t])\mathbf{1}_{\{U_{i:n}\leqslant t\}}|U_{i:n}=u)\\ &= \mathbb{E}(N_{u,V}([u,t])\mathbf{1}_{\{u\leqslant t\}}|U_{i:n}=u)\\ &= \mathbb{E}(N_{u,V}([u,t])\mathbf{1}_{\{u\leqslant t\}})\end{aligned}$$

due to independence between $U_{i:n}$ and the renewal process with inter-arrivals distributed as V.

We derive

$$g(u)=\mathbb{E}(N_V([0,t-u])\mathbf{1}_{\{u\leqslant t\}})=\rho_V(t-u)\mathbf{1}_{\{u\leqslant t\}}=\rho_V((t-u)^+)$$

and

$$\mathbb{E}(N_{U_{i:n},V}([U_{i:n},t])\mathbf{1}_{\{U_{i:n}\leqslant t\}})=\mathbb{E}(\rho_V((t-U_{i:n})^+))$$

due to (2). Hence, we obtain the first result using (1), from where we easily derive the second one, after reduction. □

We now conclude this section with the following theorem, which is a direct consequence of Lemmas 2 and 3 using $C_K([0,t])=\mathrm{CE}_K([0,t])+\mathrm{CR}_K([0,t])$.

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Theorem 4
Let $t\geqslant 0$. For $1\leqslant K\leqslant n$, we have

$$C_K([0,t])=\sum_{i=1}^{K}[(r+c_f)(F_{U_{i:n}}(t)+\mathbb{E}(\rho_V((t-U_{i:n})^+)))+v\mathbb{E}(U_{i:n}^t)]$$

$$+(n-K)[c_p F_{U_{K:n}}(t)+(r+c_f)\mathbb{E}(\rho_V((t-U_{K:n})^+))+v\mathbb{E}(U_{K:n}^t)]+n\eta t$$

and setting

$$g_K(t):=\frac{1}{c_p}(C_{K+1}([0,t])-C_K([0,t])) \tag{3}$$

for $0\leqslant K\leqslant n-1$, we have

$$\begin{aligned} g_K(t)=(a-1)F_{U_{K+1:n}}(t)+(n-K)[b\mathbb{E}(U_{K+1:n}^t-U_{K:n}^t)-(F_{U_{K:n}}(t)-F_{U_{K+1:n}}(t))\\ -a\mathbb{E}(\rho_V((t-U_{K:n})^+-\rho_V((t-U_{K+1:n})^+)))] \end{aligned} \tag{4}$$

for $1\leqslant K\leqslant n-1$ and

$$g_0(t)=(a-1)F_{U_{1:n}}(t)+n[b\mathbb{E}(U_{1:n}^t)-\bar{F}_{U_{1:n}}(t)-a\mathbb{E}(\rho_V(t)-\rho_V((t-U_{1:n})^+))]-\frac{r}{c_p}$$

4. COMPARISON BETWEEN STRATEGIES $0,1,\ldots,n$ IN LONG-TIME RUN

In order to understand what are the possible minima of $(C_K([0,t]))_{0\leqslant K\leqslant n}$ for t fixed, we are interested here in the comparison of $C_K([0,t])$'s in long-time run, namely when $t\to+\infty$. We first compute the limit of $g_K(t)$ when $t\to+\infty$, where $g_K(t)$ is defined by (3).

Proposition 5
Assume the distribution of V to be non-arithmetic and $\mathbb{E}(U_i)<+\infty$ for all $1\leqslant i\leqslant n$. Setting $g_K(\infty):=\lim_{t\to+\infty} g_K(t)$ for all $0\leqslant K\leqslant n-1$, we then have

$$g_K(\infty)=a-1+\left(b-\frac{a}{\mathbb{E}(V)}\right)(n-K)\mathbb{E}(U_{K+1:n}-U_{K:n})<+\infty$$

for all $1\leqslant K\leqslant n-1$ and

$$g_0(\infty)=\frac{c_f}{c_p}-1+\left(b-\frac{a}{\mathbb{E}(V)}\right)n\mathbb{E}(U_{1:n}-U_{0:n})<+\infty$$

where we set $U_{0:n}:=0$.

Proof
We start from (4). Owing to Blackwell's theorem and assumption on V, we know

$$\lim_{t\to+\infty}(\rho_V((t-u)^+)-\rho_V((t-v)^+))=\lim_{t\to+\infty}(\rho_V(t-u)-\rho_V(t-v))=\frac{v-u}{\mathbb{E}(V)}$$

for all $0\leqslant u\leqslant v$ with $1/\mathbb{E}(V)=0$ if $\mathbb{E}(V)=+\infty$.

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Let $0\leqslant K\leqslant n-1$. As $0\leqslant U_{K:n}<U_{K+1:n}<+\infty$ a.e. due to $U_i<+\infty$ a.e., we derive

$$\lim_{t\to+\infty}(\rho_V((t-U_{K:n})^+)-\rho_V((t-U_{K+1:n})^+))=\frac{U_{K+1:n}-U_{K:n}}{\mathbb{E}(V)}$$

and the result, using $\lim_{t\to+\infty} F_{U_{K:n}}(t)=1$ and $\lim_{t\to+\infty}\mathbb{E}(U^t_{K:n})=\mathbb{E}(U_{K:n})<+\infty$ by monotone theorem (the same for $K+1$). □

A first consequence is that, if $b-a/\mathbb{E}(V)\geqslant 0$ or alternatively $v\geqslant(r+c_f)/\mathbb{E}(V)$, we then have $g_K(\infty)\geqslant 0$ for all $0\leqslant K\leqslant n-1$ (we recall that $a\geqslant 1$ and $c_f\geqslant c_p$). Consequently, if $v\geqslant(r+c_f)/\mathbb{E}(V)$, the best strategy among $0,\dots,n$ in long-time run is strategy 0. Such a result conforms to the intuition: indeed, let us recall that v stands for the additional energy consumption rate for the old-type units compared with the new-type ones; also, we observe that $(r+cf)/\mathbb{E}(V)$ is the cost rate per unit time for replacements due to failures among new-type components in long-time run. Then, the result means that if replacements of new-type components due to failures are less costly per unit time than the benefit due to a lower consumption, it is better to replace old-type components by new-type ones as soon as possible.

Now, we have to look at the case $b-a/\mathbb{E}(V)<0$ and for that, we have to study the monotony of

$$D_K:=(n-K)(U_{K+1:n}-U_{K:n})$$

with respect to K, where D_K is the Kth normalized spacing of the order statistics $(U_{1:n},\dots,U_{n:n})$, see, for example, [23, 24].

With that aim, we now put some assumption on the distributions of the residual lifetimes of the old-type components at time $t=0$ (U_i for $1\leqslant i\leqslant n$), from where we shall induce properties for $(D_K)_{0\leqslant K\leqslant n}$. Without any additional knowledge, a natural hypothesis is to assume that the ith unit has already been replaced a large number of times. Assuming such replacement times for the ith unit to make a renewal process with inter-arrival times distributed as some $U^{(0)}$ (independent of i), the residual life at time 0 for the ith unit may then be considered as the waiting time until next arrival for a stationary renewal process with inter-arrivals distributed as $U^{(0)}$. Such a waiting time is known to admit as p.d.f. the function $f_U(t)$ such that

$$f_U(t)=\frac{\bar{F}_{U^{(0)}}(t)}{\mathbb{E}(U^{(0)})}\mathbf{1}_{\mathbb{R}^+}(t) \tag{5}$$

in the case $0<\mathbb{E}(U^{(0)})<+\infty$. We consequently assume that U_i admits such function $f_U(t)$ as p.d.f. in the following.

Under such assumption, it is well known that U is exponentially distributed if and only if $U^{(0)}$ has the same property. Similarly, it is also well known that $D_1,\dots,D_n$ are i.i.d. if and only if the underlying distribution for U is exponential. As a consequence, $D_1,\dots,D_n$ are i.i.d. if and only if the distribution of $U^{(0)}$ is exponential. In particular, the sequence $(g_K(\infty))_{1\leqslant K\leqslant n}$ is constant when $U^{(0)}$ is exponentially distributed; hence, $(g_K(\infty))_{1\leqslant K\leqslant n}$ has a constant sign. This implies that the optimal strategy among $1,\dots,n$ can be only strategy 1 or n in long-time run, result already proved for a finite horizon in [19].

When $U^{(0)}$ is not exponentially distributed; hence neither U, we now use a result from [23] according to which, if $U_1,\dots,U_n$ are i.i.d. increasing failure rate (IFR) r.v. with $F_{U_i}(0)=0$, then the successive normalized spacings $(D_K)_{0\leqslant K\leqslant n}$ associated with U_i's are stochastically decreasing.

We first prove that the IFR property of $U^{(0)}$ is passed on to U.

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Lemma 6
Assume $U^{(0)}$ to be a non-negative IFR r.v. with $0<\mathbb{E}(U^{(0)})<+\infty$ and U to be a continuous r.v. with p.d.f. given by (5). Then U is also IFR.

Proof
As U admits some density function, we may use its associated hazard rate function $h_U(t)$ to prove its IFR property, with

$$h_U(t)=\frac{f_U(t)}{\bar{F}_U(t)}=\frac{\bar{F}_{U^{(0)}}(t)}{\int_t^{+\infty}\bar{F}_{U^{(0)}}(u)\,\mathrm{d}u}=\frac{\bar{F}_{U^{(0)}}(t)}{\int_0^{+\infty}\bar{F}_{U^{(0)}}(t+u)\,\mathrm{d}u}$$

for $t\geqslant 0$ such that $\bar{F}_U(t)>0$, or equivalently for $t\geqslant 0$ such that $\bar{F}_{U^{(0)}}(t)>0$. We then have to prove that $h_U(t)$ is increasing with t on $\{t\geqslant 0:\bar{F}_{U^{(0)}}(t)>0\}$. Now, let $0\leqslant t_1<t_2$ be such that $\bar{F}_{U^{(0)}}(t_2)>0$ (hence $\bar{F}_{U^{(0)}}(t_1)>0$). We have to prove that $h_U(t_1)\leqslant h_U(t_2)$ or alternatively that

$$\frac{\bar{F}_{U^{(0)}}(t_1)}{\int_0^{+\infty}\bar{F}_{U^{(0)}}(t_1+u)\,\mathrm{d}u}\leqslant\frac{\bar{F}_{U^{(0)}}(t_2)}{\int_0^{+\infty}\bar{F}_{U^{(0)}}(t_2+u)\,\mathrm{d}u}$$

This may also be expressed as

$$\int_0^{+\infty}\left(\frac{\bar{F}_{U^{(0)}}(t_2+u)}{\bar{F}_{U^{(0)}}(t_2)}-\frac{\bar{F}_{U^{(0)}}(t_1+u)}{\bar{F}_{U^{(0)}}(t_1)}\right)\mathrm{d}u\leqslant 0$$

which is true and allows to conclude, because $U^{(0)}$ is assumed to be IFR so that $\bar{F}_{U^{(0)}}(t+u)/\bar{F}_{U^{(0)}}(t)$ is decreasing in t for all $u>0$. $\square$

We are now ready to state our result.

Theorem 7
If $b-a/\mathbb{E}(V)\geqslant 0$, the optimal strategy among $0,\dots,n$ in long-time run is strategy 0.
In the case $b-a/\mathbb{E}(V)<0$, assume that $U^{(0)}$ is a non-negative IFR r.v. with $0<\mathbb{E}(U^{(0)})<+\infty$ and that U is a continuous r.v. with p.d.f. given by (5). Assume beside that $U^{(0)}$ is not exponentially distributed. The sequence $(\mathbb{E}(D_K))_{0\leqslant K\leqslant n-1}$ is then strictly decreasing, and setting

$$c:=\frac{a-1}{a/\mathbb{E}(V)-b}\quad\text{and}\quad d:=\frac{c_f/c_p-1}{a/\mathbb{E}(V)-b}\leqslant c$$

one of the following cases occurs:

- if $c\leqslant\mathbb{E}(D_{n-1})$: the optimal strategy among $0,\dots,n$ in long-time run is strategy n;
- if $c>\mathbb{E}(D_1)$:
  - if $d>\mathbb{E}(D_0)$: the optimal strategy among $0,\dots,n$ in long-time run is strategy 0;
  - if $d\leqslant\mathbb{E}(D_0)$: the optimal strategy among $0,\dots,n$ in long-time run is strategy 1;
- if $\mathbb{E}(D_{K_0})<c\leqslant\mathbb{E}(D_{K_0-1})$ for some $2\leqslant K_0\leqslant n-1$: the optimal strategy among $0,\dots,n$ in long-time run is strategy K_0.

Remark 8
The results of the previous theorem are valid as soon as $U_1, \ldots, U_n$ are i.i.d. IFR. Apart from the natural assumption on U_i's described just above, another possibility to meet with this assumption would be that the old-type components had been put into activity simultaneously (with IFR lifetimes).

Proof
As already mentioned just after Proposition 5, the case $b - a/\mathbb{E}(V) \geqslant 0$ is clear and we consider only the case $b - a/\mathbb{E}(V) < 0$. In that case, we have for $1 \leqslant K \leqslant n-1$:

$$(g_K(\infty) < 0) \Longleftrightarrow (c < \mathbb{E}(D_K))$$

and $(g_0(\infty) < 0) \Longleftrightarrow (d < \mathbb{E}(D_0))$.

Owing to assumptions on $U^{(0)}$ and U, we know from Lemma 6 that $U_1, \ldots, U_n$ are i.i.d. IFR r.v. with $F_{U_i}(0) = 0$. Using the recalled result from [23], we derive that $(D_K)_{0 \leqslant K \leqslant n}$ is stochastically decreasing with K so that $(\mathbb{E}(D_K))_{0 \leqslant K \leqslant n}$ is decreasing, and actually strictly decreasing as soon as $U^{(0)}$ is not exponentially distributed.

Then, if $c < \min_{1 \leqslant K \leqslant n-1} \mathbb{E}(D_K) = \mathbb{E}(D_{n-1})$, we know that $g_K(\infty) < 0$ for all $1 \leqslant K \leqslant n-1$ and $S^{\text{opt}}_{1:n} = n$ with clear notations. Beside, $d \leqslant c < \mathbb{E}(D_{n-1}) < \mathbb{E}(D_0)$, so that $S^{\text{opt}}_{0:1} = 1$. We derive $S^{\text{opt}}_{0:n} = n$.

The case $c > \mathbb{E}(D_1)$ is similar and omitted.

Now assume $\mathbb{E}(D_{K_0}) < c \leqslant \mathbb{E}(D_{K_0 - 1})$ for some $2 \leqslant K_0 \leqslant n-1$. We then have

$$g_1(\infty) < \cdots < g_{K_0-1}(\infty) \leqslant 0 < g_{K_0}(\infty) < g_{K_0+1}(\infty) < \cdots < g_{n-1}(\infty)$$

and consequently $S^{\text{opt}}_{1:n} = K_0$. Beside, $d \leqslant c \leqslant \mathbb{E}(D_{K_0-1}) < \mathbb{E}(D_0)$ and $g_0(\infty) < 0$. We derive $S^{\text{opt}}_{0:1} = 1$ and $S^{\text{opt}}_{0:n} = K_0$. $\square$

Recall that we had proved in [19] the following 'dichotomy' property: in the case of constant failure rates, only strategies purely preventive (0), nearly purely preventive (1) or purely corrective (n) can be optimal for finite horizon. We now know from Theorem 7 that such a property is not valid anymore in the case of general failure rates, at least for infinite horizon and consequently for large t.

For smaller t, some results from the long-time run are still valid. In this way, under the assumptions of Lemma 6, setting $(U^t)_{K:n}$ to be the Kth order statistic of $(U_1^t, \ldots, U_n^t)$ one may prove that $\mathbb{E}(D_K^t) := (n-K)\mathbb{E}(U^t_{K+1:n} - U^t_{K:n}) = (n-K)\mathbb{E}((U^t)_{K+1:n} - (U^t)_{K:n})$ is still decreasing, because U^t is still IFR. However, this observation seems to be quite insufficient to lead the study of the monotony of $(g_K(t))_{0 \leqslant K \leqslant n}$ up to its end. Consequently, it seems difficult to find the optimal strategy from a theoretical point of view for smaller t.

We consequently observe numerically in the following section the behavior of $(C_K([0,t]))_{0 \leqslant K \leqslant n}$ with respect to K for t 'not too large'.

5. NUMERICAL EXPERIMENTS

All the computations are here made with Matlab.

We assume that $U^{(0)}$ is Weibull distributed ($W(\alpha_1, \beta_1)$) with survival function:

$$\bar{F}_{U^{(0)}}(x) = \mathrm{e}^{-\alpha_1 x^{\beta_1}}$$

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where $\alpha_1 > 0$ and $\beta > 1$ ($U^{(0)}$ is IFR). We then take U with p.d.f. given by (5):

$$f_U(x) = \frac{\mathrm{e}^{-\alpha_1 x^{\beta_1}}}{\mathbb{E}(U)} \mathbf{1}_{\mathbb{R}_+}(x) = \frac{\alpha_1^{1/\beta_1}}{\Gamma\left(1+\dfrac{1}{\beta_1}\right)} \mathrm{e}^{-\alpha_1 x^{\beta_1}} \mathbf{1}_{\mathbb{R}_+}(x)$$

so that

$$F_U(x) = \int_0^x \frac{\alpha_1^{1/\beta_1}}{\Gamma\left(1+\dfrac{1}{\beta_1}\right)} \mathrm{e}^{-\alpha_1 u^{\beta_1}} \,\mathrm{d}u = \Gamma_{\mathrm{inc}}\left(\alpha_1 x^{\beta_1}, \frac{1}{\beta_1}\right) \tag{6}$$

for all $x \geqslant 0$ after reduction, where Γ_{inc} is the incomplete Gamma function. This function is implemented in Matlab, so that F_U and $\bar{F}_U$ are easy to compute from (6).

We derive $F_{U_{K:n}}$ using

$$\begin{aligned} F_{U_{K:n}}(x) &= \int_0^{F_U(x)} \frac{n!}{(K-1)!(n-K)!} t^{K-1}(1-t)^{n-K} \,\mathrm{d}t \\ &= I_{F_U(x)}(K, n-K+1) \end{aligned}$$

for $1 \leqslant K \leqslant n$, where $I_x(n_1, n_2)$ is the incomplete Beta function (also implemented in Matlab), see, for example, [25] for the results on order statistics used in this section.

We also use

$$\bar{F}_{U_{K+1:n}}(t) - \bar{F}_{U_{K:n}}(t) = \binom{n}{K} F_U^K(t) \bar{F}_U^{n-K}(t)$$

from where we derive $\mathbb{E}(U_{K+1:n}^t - U_{K:n}^t)$ due to

$$\mathbb{E}(U_{K+1:n}^t - U_{K:n}^t) = \int_0^t (\bar{F}_{U_{K+1:n}}(u) - \bar{F}_{U_{K:n}}(u)) \,\mathrm{d}u$$

for $0 \leqslant K \leqslant n-1$ (we recall $U_{0:n} := 0$).

We finally compute $\mathbb{E}(\rho_V((t-U_{K:n})^+))$ with

$$\begin{aligned} \mathbb{E}(\rho_V((t-U_{K:n})^+)) &= \int_0^t \rho_V(t-u) \,\mathrm{d} f_{U_{K:n}}(t) \\ &= n \binom{n-1}{K-1} \int_0^t \rho_V(t-u) F_U^{K-1}(u) \bar{F}_U^{n-K}(u) f_U(u) \,\mathrm{d}u \end{aligned}$$

where the renewal function ρ_V is computed via the algorithm from [26], which here ensures a relative precision less than 3.5×10^{-4} (with the numerical data given just below).

We assume V to be Weibull distributed $W(\alpha_2, \beta_2)$ and we take

$$\alpha_1 = \frac{1}{10^3}, \quad \beta_1 = 2.8, \quad \alpha_2 = \frac{1}{2 \times 10^3}, \quad \beta_2 = 3.2$$

so that $\mathbb{E}(U^{(0)}) \simeq 10.50$ and $\mathbb{E}(V) \simeq 9.63$.

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Table I. Optimal strategy according to t and v.

$t\backslash v$	0	0.04	0.05	0.06	0.07	0.08	0.09	0.095	0.0975	0.1	0.2
5	10	10	10	10	10	10	10	10	10	10	0
10	10	10	10	10	3	3	2	2	2	2	0
15	10	7	7	7	6	6	6	5	5	5	0
20	10	9	9	8	8	4	2	1	1	1	0
25	10	8	8	7	6	5	4	3	3	2	0
30	10	9	8	7	7	5	3	2	2	1	0
35	10	9	8	7	6	5	4	3	2	1	0
100	10	9	8	7	6	5	4	3	2	1	0
∞	10	9	8	7	6	5	4	3	2	1	0

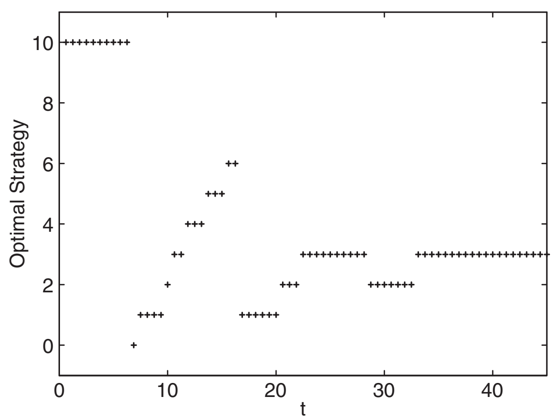

Figure 1. Optimal strategy with respect to t for $v=0.095$.

We also take

$$n=10, \quad \eta=0, \quad c_p=5, \quad c_f=7, \quad r=4$$

For finite horizon, the optimization on K is simply made by computing all $C_K([0,t])$ for $K=0,\ldots,n$ and finding the smallest. For infinite horizon, Theorem 7 is used.

The optimal strategy is given in Table I for different values of v and t, as well as the asymptotic results.

We can see in such a table that the optimal strategy is quickly stable with increasing t. More precisely, the optimal strategy for a finite horizon t is the same as the optimal strategy in long-time run as soon as t is greater than about 3.5 mean lengths of life of a component. (Note that we here have $\mathbb{E}(U^{(0)})\simeq\mathbb{E}(V)\simeq 10$.)

We now plot in Figure 1 the optimal strategy with respect to t for some fixed v ($v=0.095$). We can see in such a figure that the behavior of K^{opt} (optimal K) with increasing t is not regular. There is consequently no hope to get any clear characterization of K^{opt} with respect to the different parameters in finite horizon as we had in the exponential case in [19] and as we have here in infinite horizon (Theorem 7).

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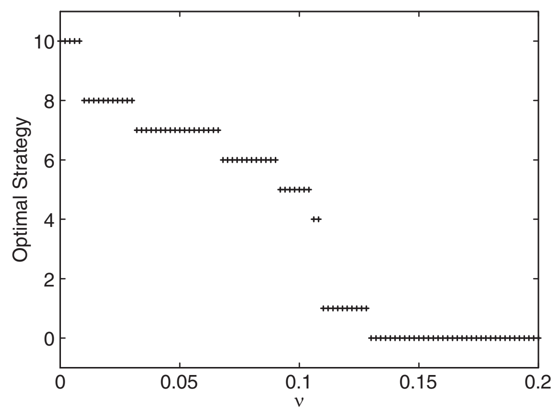

Figure 2. Optimal strategy with respect to v for $t=15$.

We finally plot K^{opt} for t fixed ($t=15$) with respect to v, which shows that K^{opt} may vary a lot changing one single parameter (Figure 2).

6. CONCLUSION

In conclusion, the behavior of $(C_K([0,t]))_{0\leqslant K\leqslant n}$ with K is much less regular in the present case of general failure rates than in the case of constant failure rates as in [18, 19]. Also, the main result from [19], which shows that the optimal strategy could be only strategy 0, 1 or n, is here false. It does not seem possible here to give clear conditions on the data to foretell which strategy is optimal in finite horizon as was done in [19]. We, however, obtained such conditions in long-time run. A few numerical experiments (adding to those given here) seem to indicate that the optimal strategy in long-time run actually is quickly optimal, namely for t not that large. The results for long-time run then seem to give a good indication for the choice of the best strategy, even for t not very large.

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